Diploma Programme
Programme du diplôme
Programa del Diploma

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Diploma Programme
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Mathematics: applications and interpretation
Higher level
Paper 3

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **both** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

This question uses differential equations to model the maximum velocity of a skydiver in free fall.

In 2012, Felix Baumgartner jumped from a height of $40\,000\,\mathrm{m}$. He was attempting to travel at the speed of sound, $330\,\mathrm{m\,s^{-1}}$, whilst free-falling to the Earth.

Before making his attempt, Felix used mathematical models to check how realistic his attempt would be. The simplest model he used suggests that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g$$

where $v\,\mathrm{m\,s^{-1}}$ is Felix's velocity and $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity. The time since he began to free-fall is t seconds and the displacement from his initial position is s metres.

Throughout this question, the direction towards the centre of the Earth is taken to be positive and v is a positive quantity.

When $s = 0$, it is given that Felix jumps with an initial velocity $v = 10$.

(a) (i) Use the chain rule to show that $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$. [1]

(ii) Assuming that g is a constant, solve the differential equation $v\frac{\mathrm{d}v}{\mathrm{d}s} = g$ to find v as a function of s. [4]

(iii) Using $g = 9.8$, determine whether the model predicts that Felix will succeed in travelling at the speed of sound at some point before $s = 40\,000$. Justify your answer. [3]

(b) To test the model

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\,,$$

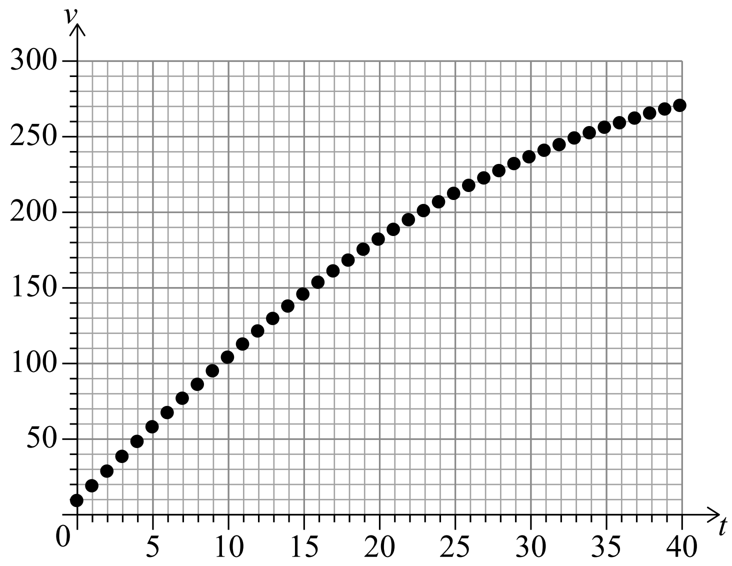
Felix conducted a trial jump from a lower height, and data for v against t was found.

(i) If the model is correct, describe the shape of the graph of v against t. [2]

(This question continues on the following page)

(Question 1 continued)

Felix's data are plotted on the following graph.



(ii) Use the plot to comment on the validity of the model in part (a). [1]

(c) An improved model considers air resistance, using

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2$$

where k is a positive constant. You are reminded that initially $s = 0$ and $v = 10$.

(i) By using $\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}$, solve the differential equation to find v in terms of s, g and k. You may assume that $g - kv^2 > 0$. [5]

Felix uses the graph of v against t shown in part (b) to estimate the value of k.

(ii) The gradient is estimated to be 9.672 when $v = 40$. Taking g to be 9.8, use this information to show that Felix found that $k = 8 \times 10^{-5}$. [2]

(iii) Hence, find the value of v predicted by this model, as s tends to infinity. [2]

(iv) Find the upper bound for the velocity according to this model, given that $0 < s \leq 40\,000$. Give your answer to four significant figures. [2]

(This question continues on the following page)

(Question 1 continued)

The assumption that the value of g is constant is not correct. It can be shown that

$$g = \frac{3.98 \times 10^{14}}{\left(6.41 \times 10^{6} - s\right)^{2}}.$$

Hence, the new model is given by

$$v\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{3.98 \times 10^{14}}{\left(6.41 \times 10^{6} - s\right)^{2}} - \left(8 \times 10^{-5}\right)v^{2}.$$

When $s = 0$, it is known that $v = 10$.

(d) Use Euler's method with a step length of 4000 to estimate the value of v when $s = 40\,000$. [4]

(e) After Felix completed his record-breaking jump, he found that the answer from part (d) was not supported by data collected during the jump.

(i) Suggest **one** improvement to the use of Euler's method which might increase the accuracy of the prediction of the model. [1]

(ii) Suggest **one** factor **not** explicitly considered by the model in part (d) which might lead to a difference between the model's prediction and the data collected. [1]

2. [Maximum mark: 27]

This question is about applying ideas from logarithms, calculus and probability to an unfamiliar mathematical theory called information theory.

Claude Shannon developed a mathematical theory called information theory to measure the information gained when random events occur. He defined the information, I, that is gained when an event with probability p occurs as

$$I = -\ln p$$

where $0 < p \leq 1$. For example, no information is gained ($I = 0$) when an event is certain to occur ($p = 1$).

(a) (i) Sketch the graph of $I = -\ln p$, for $0 < p \leq 1$, labelling all axes intercepts and asymptotes. [3]

(ii) Show, using calculus, that I is a decreasing function of p. [3]

(iii) Interpret what "I is a decreasing function of p" means in the given context. [1]

(b) A computer selects at random an integer x from 1 to 10, inclusive. Each outcome is equally likely.

Alessia is trying to determine the value of x and asks if x is odd.

(i) Write down the probability that x is odd. [1]

(ii) Alessia is told that x is odd. Find how much information Alessia gains. [2]

The computer then selects at random an integer y from 1 to 10, inclusive.
Each outcome is equally likely.

Daniel is trying to determine the value of y and asks if y is 7. He is told that it is not 7.

(iii) Find how much information Daniel gains. [2]

(This question continues on the following page)

(Question 2 continued)

If a random variable has n possible outcomes with probabilities $p_1, p_2 \ldots p_n$, then the expected information gained, $\mathrm{E}(I)$, is defined as

$$\mathrm{E}(I)=\sum_{r=1}^{n}-p_r \ln p_r .$$

(c) For the integer guessing game described in part (b), when Daniel asks if y is 7, there are two possible outcomes: "y is 7" or "y is not 7".

(i) Show that the expected information gained by Daniel is 0.325, correct to three significant figures. [2]

(ii) Alessia asks if x is odd. Show that her expected information gained is greater than Daniel's expected information gained. [2]

Information theory can be applied to a variety of situations.

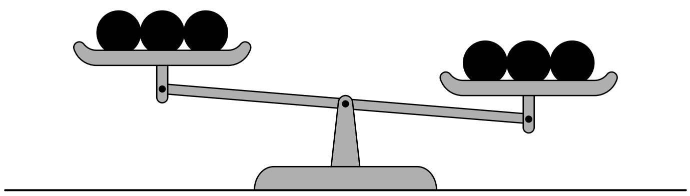
(d) When a coin is flipped, the outcome is either heads or tails. The coin may be biased. Let p be the probability of the outcome being heads.

(i) Find, in terms of p, the information gained when the outcome is tails. [1]

(ii) Find, in terms of p, the expected information gained when the coin is flipped once. [1]

(iii) Hence, find the value of p when the expected information gained is maximized. [2]

A famous puzzle uses 12 balls which appear identical. 11 have the same weight, but one is either lighter or heavier than the others. A pair of scales can be repeatedly used to compare the weights of different combinations of the balls.



The outcome of each weighing can be "balanced", "left-hand side heavier" or "right-hand side heavier". The aim of the puzzle is to identify the ball which is the different weight, and whether it is heavier or lighter than the others, in as few weighings as possible.

(This question continues on the following page)

(Question 2 continued)

(e) Angela wants to decide how many balls should be compared to each other in the **first weighing**. She produces the following table to help plan her strategy.

Number of balls on each side	Probability of balanced	Probability of left-hand side heavier	Probability of right-hand side heavier	Expected information, $\mathrm{E}(I)$ (3 decimal places)
1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	0.566
2	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	z
3	x	y	y	1.040
4	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1.099
5	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{5}{12}$	1.028
6	0	$\frac{1}{2}$	$\frac{1}{2}$	0.693

(i) Find the value of x. Justify your answer. [2]

(ii) Find the value of y. [2]

(iii) Find the value of z. [2]

(iv) Use the table to suggest the best choice for Angela's first weighing. Justify your answer. [1]